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Investigation of turbulent convection under a rotational constraint

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Turbulent convection for a rotating layer of fluid heated from below is studied in this paper. The boundaries of the fluid layer are taken to be free. The underlying principle used is the Malkus hypothesis that the flow tends to transport the maximum amount of heat possible, subject to certain constraints. By taking the Prandtl number to be infinite, a linear differential constraint and an integral constraint are used. The variational problem that follows then depends on two dimensionless parameters, the Taylor number T and the Rayleigh number R.

Asymptotic analysis for the turbulent regime shows that the flow arranges itself so as to tend to offset the stabilizing effect of the rotational constraint, at least in so far as the heat flux is concerned. The dimensionless heat flux, or the Nusselt number, has in general different dependence on T and R, depending on the particular region in the parameter space. For $T \leqslant O(R)$, the flow is essentially non-rotating. For $O(R) \leqslant T \leqslant O(R^{\frac{4}{3}})$, the flow will always have finitely many horizontal wavenumbers, though the total number of modes increases as T increases in this region. For $O(R^{\frac{4}{3}}) \leqslant T \leqslant O(R^{\frac{3}{2}})$, the Nusselt number has a functional dependence proportional to R^3/T^2, having essentially infinitely many horizontal modes as both R and T increase indefinitely in this region. The last expression is particularly interesting, as it agrees qualitatively with results in finite-amplitude laminar convection. It is also linearly dependent on the layer thickness, as one might expect from dimensional argument. It is suggested that, in the context of the maximum principle, the result in this region of the parameter space may be applicable as well to the same fluid layer with rigid boundaries through the existence of an Ekman layer that is thinner than the thermal layer.

1. Introduction

The closure problem in the study of turbulent shear flows is well known. The problem arises because of the nonlinearity in the Navier–Stokes equation so that the equation for the nth statistical moment of the flow quantities depends on the $(n+1)$th moment resulting in a hierarchy of coupled equations. To reduce the infinite system to a finite one, various closure schemes are used. They amount to the replacing of the equations of motion by some weaker constraints. One can therefore legitimately ask if some weak constraints can be found that will bypass the difficulty of having to deal with the nonlinear terms directly, yet give a reasonable description of the mean quantities. The Malkus theory of statistically steady turbulence serves just such a purpose. In essence, this is accomplished by establishing an extremum principle subject to some constraints weaker than the equations of motion. In its original form, the principle of maximum dissipation was offered as plausible (Malkus 1954, 1956).

Townsend (1961) classified the Malkus assumptions as falling under two categories: kinematic and dynamical. Using just the kinematic assumptions alone, he succeeded in obtaining the approximate velocity profile by considering the best way to approximate the asymptotic distribution with a finite series giving non-positive values for the mean vorticity gradient. By using the same kinematic assumptions, Nihoul (1966) was able to extend the Malkus theory, and obtained the mean velocity profile in an MHD channel flow.

With a different approach, Howard (1963) recast the Malkus theory on statistically steady turbulent convection in the form of a variational problem. He placed the emphasis on the aspect of maximum dissipation which in that case is equal to maximum heat transport. As constraints, Howard used the two first integrals, called power integrals, of the equations of motion. Subsequently, Busse (1968, 1969) extended Howard's formal technique to allow a broader class of solutions, and applied it to both turbulent convection and turbulent shear flows. Aiming at improving the upper bound on the heat flux, Chan (1971) applied the technique to the study of turbulent convection when the Prandtl number was taken to be infinite. The effect is to replace the first power integral by a linear differential constraint which implies the former. In this sense, then, it tightens up the field of competitors for the variational problem, thus giving rise to a better bound when compared with experiments.

While the upper-bound solution in either form does not satisfy the full equation of motion, and must therefore not be confused with the real flow, one could nonetheless ask whether or not the upper-bound solution bears any qualitative or quantitative relation to the real flow, beyond merely providing an upper bound to the heat flux. For this purpose, we shall examine the idea behind some of the closure schemes in the theory of statistical turbulence. For example, in the quasi-normal approximation, the hierarchy of moment equations is closed by assuming a relation between the fourth- and the second-order cumulants as if the joint probability distribution is normal. Alternatively, in the third-order cumulant discard scheme, the closure is achieved by discarding cumulants of third order and higher. Whatever closure scheme one uses, the hope is to replace an infinite system by a finite one, and that the solution from the finite system will be close enough to the real flow. In fact, the relation between the solution space of the full equations of motion and the solution space under any closure scheme is not clear. It is therefore entirely possible that the approximating solution, even if it is close to the real flow, is itself not a member of the exact solution space, as illustrated in figure 1. Indeed, the quasi-normal approximation is found to possess non-physical properties such as a negative energy spectrum (Ogura 1969), and is therefore not a real flow. While the general feature as

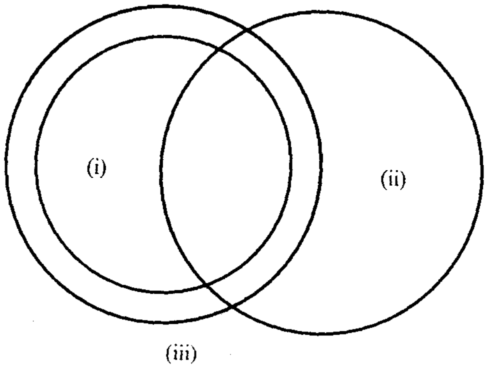

FIGURE 1. Solution space (i) of the full equations of motion, (ii) under a closure scheme, (iii) of the upper-bound approach (under power-integral constraints).

exhibited in figure 1 also holds for the third-order cumulant discard scheme, Herring (1963) showed that it is equivalent to the quasi-linear or the mean-field approximation, which includes only the interaction between the mean and the fluctuating fields, but ignores the interaction among the fluctuating quantities. In this approximation, however, the energy balance of the real flow is retained.

The upper-bound solution with power-integral constraints, on the other hand, attempts to approximate the real flow from a different track. It is clear that the power integrals are part of the infinite system of moment equations (e.g. the so-called first power integral can be obtained by considering the correlation between the velocity $\mathbf{u}(\mathbf{x}, t)$ and the momentum equation at the same $\mathbf{x}$ and t, and then averaging it over the layer). Incompressibility of the fluid now assures that the effect of the nonlinear term will not be included, thus bypassing the closure problem. Using these power integrals instead of the infinite system as a constraint therefore enlarges the class of functions from which we seek the upper-bound solution. This class of functions now contains the set of solutions to the full equations of motion (as also illustrated in figure 1). This is the price we pay for bypassing the closure problem. However, the approximation thus found has the advantage of retaining the energy balance of the real flow. Furthermore, it could at least be said that this approach seeks an approximation from a class of functions which contains all of the real flows. Whether or not the upper-bound solution is a reasonable approximation to the real flow really depends on, among other things, how much larger the set (iii) is compared with the set (i). In fact, Howard suggested that 'it is not unreasonable to suppose that the successive imposition of more and more integral consequences of the Boussinesq equations as constraints on the problem of maximum heat transport will give a sequence of problems whose solutions converge in some useful sense to the solution of the problem with the full Boussinesq equations as constraints'. In other words, it is reasonable to view the upper-bound solution under the power-integral constraint as a first step toward obtaining a useful approximation.

Similarly, it is suggested that the infinite Prandtl number treatment can be viewed in the same light. To be sure, the relation between the solution space of the full equations of motion and that of the infinite Prandtl number treatment is not clear. In any event, Chan (1971) showed that the optimal solutions (with $\sigma = \infty$) have the same structure as the mean-field solutions and in fact satisfy the mean-field equations. The only difference lies in the proportionality constants that will be determined by a maximal principle for the optimal solutions and presumably by some stability criterion for the mean-field equations. It seems reasonable, therefore, to view the solution as at least in some sense a first step toward the upper-bound solution using the full equations of motions as constraints, despite the conceptual difficulty that the solution resulting from the weaker constraints does not even satisfy the full equations of motion and the fact that the upper bound thus obtained is only valid for high Prandtl number flows. In fact, the good agreement, both qualitative and quantitative, with experiment for the non-rotating case (Chan 1971) makes it difficult not to view the upper bound solution (with $\sigma = \infty$) as a first-order approximation to the real flow. While the author does not necessarily advocate that the upper-bound solutions (in either form) be taken as an acceptable approximation to the real flow, he nonetheless feels, in view of the above explanations, that no great harm will be done if we look at the upper-bound solution as representing the real flow, keeping in mind that perhaps the discussion is valid only when the Prandtl number of the fluid is high enough or when the interactions among the fluctuating fields are negligible.

At any rate, since the first power integral is a statement of energy balance, it cannot differentiate additional constraints on the flow as long as those constraints are energetically inactive, e.g. a rotational constraint. The infinite Prandtl number treatment, on the other hand, is particularly suitable as long as those energetically inactive constraints appear linearly in the momentum equation. Taking the Prandtl number to be infinite, Chan (1972) extended the study on free convection to the case with a rotational constraint. It was shown that the optimal solution was asymptotic to the finite-amplitude result of Veronis (1959). The case of turbulent convection allowing only one horizontal scale was also obtained.

In this paper, we study the problem of turbulent convection under a rotational constraint, again taking the Prandtl number to be infinite, but allowing the solution to have as many horizontal scales as possible. Each horizontal mode, characterized by a horizontal wavenumber α, corresponds physically to a certain horizontal scale of motion (e.g. the horizontal scale of the eddies). The different scales of motion are of course coupled because the problem is nonlinear. By considering only the boundary-layer solution when the parameters are large, however, it is possible to handle the nonlinear coupling which in that case occurs only between two succeeding scales of motion. In any event, the problem is characterized by two dimensionless parameters, the Taylor number T and the Rayleigh number R. The resultant dimensionless heat flux, or the Nusselt number Nu is rather complicated, depending on the relation between T and R. Briefly, the parameter space can be divided into three regions. For a weakly rotating constraint (i.e. $T < O(R)$), the effect of the rotation is not felt. For a moderately rotating system (i.e. $O(R) < T < O(R^{\frac{4}{3}})$), the primary effect of the rotation is the suppression of the number of horizontal modes (i.e. there are only finitely many horizontal modes for any given T and R in this range, even as both T and R go to infinity). For a strongly rotating system (i.e. $O(R^{\frac{4}{3}}) < T < O(R^{\frac{3}{2}})$), the flow is allowed to have infinitely many horizontal modes as both T and R go to infinity. For the last two cases, the Nusselt number is always found to be smaller than its non-rotating value, corresponding to the same R. This agrees with the conclusion of Veronis (1959) that, for high Prandtl number flows, the effect of the rotational constraint is always to suppress convective motion, so that the transport of heat is always less effective than in the non-rotating case. On the other hand, this finding differs from the experimental data of Rossby (1970), who reported that the heat flux will first increase with increasing T before it decreases. Therefore, care must be taken before applying the optimal solution or the quasi-linear approximation to a real flow. In any event, within the constraint of a high Prandtl number, the solution suggests the following physical interpretation. For a moderately rotating system, the shortest scale of motion is limited and the eddies cannot be broken down indefinitely. For a strongly rotating system, on the other hand, there is no limit to the breakdown of the eddies. It is suggested that this kind of behaviour results because the flow tends to arrange itself in such a way as to remove the effect of the rotational constraint which suppresses convection. This is done in the breaking up of the flow into small eddies which tend to cancel the suppression due to the rotation.

In the non-rotating case, simple analysis suggests (Chan 1970) that the optimal solution has only one horizontal scale, giving rise to a Nusselt number proportional to $R^{\frac{1}{3}}$. In a moderately rotating system where the suppression of heat transport is not too strong, the breakdown of the flow into small eddies must be limited, as otherwise the optimal 'flow' may result in the transport of a greater amount of heat than in the non-rotating case. On the other hand, the suppression of heat transport due to a strongly rotating constraint is so great that, no matter how small the eddies may be, it will not result in the transport of more heat than in the non-rotating case. Consequently, in its effort to offset the rotational constraint, the optimal solution results in breaking down into small eddies indefinitely. Preliminary investigation indicates that this phenomenon is found only when the boundaries are free. Apparently, it is a result of the fact that the non-rotating optimal solution allows only one eddy size, regardless of how large the Rayleigh number is (by contrast, the optimal solution for rigid–rigid boundaries has indefinitely many horizontal scales of motion as R increases). After the variational problem is formulated in §2, the asymptotic analysis leading to the above result will be discussed in §§3–5.

Though the free boundaries used in this study make it difficult to compare with experimental data, the result for a strongly rotating system is particularly interesting. The fact that the thermal layer in that case is thicker than the Ekman layer suggests that, in this region of the parameter space, the optimal solution will always behave as if the boundaries are free, even if the actual boundaries are rigid. The appropriate matching will then take place in the Ekman layer. An indication that this argument is valid may be found in the data of Rossby (1970), who reported that the onset of convection for large T is found to be at values of R lower than those predicted by the linear theory based on rigid-boundary conditions. The experimental data in fact fit those values based on free boundaries. In any event, the data of Rossby are probably obtained at values of T and R too low for asymptotic analysis, though spot checking does show that the optimal Nusselt number is in fact larger than the experimental value. These and other aspects of the analytic results are discussed in § 6.

2. The formulation of a variational problem

Consider the idealized situation of a layer of fluid, infinite in its horizontal extent, being heated from below. It is further assumed that the fluid is under the constraint of a rotation about the vertical axis. Mathematically, we take as a model the Boussinesq approximation. The model essentially treats the fluid as incompressible except for the buoyancy term in the momentum equation. Taking the thickness of the layer as d, the temperature difference as ΔT, the thermometric conductivity and the kinematic viscosity of the fluid (both being treated as constants) as κ and ν, respectively, the velocity, lengths, time and pressure can be non-dimensionalized by, respectively, κ/d, d, d^2/κ and $\rho\nu\kappa/d^2$, where ρ is the constant mean density of the fluid. In non-dimensional form, the Boussinesq equations are written as

$$\sigma^{-1}\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}.\nabla\mathbf{u}\right) = -E^{-1}\nabla P+\nabla^2\mathbf{u}+RT\hat{\mathbf{k}}+\frac{2}{E}\mathbf{u}\times\hat{\mathbf{k}}, \tag{1}$$

$$\nabla.\mathbf{u} = 0 \quad \text{and} \quad \frac{\partial T^*}{\partial t}+\mathbf{u}.\nabla T^* = \nabla^2 T^*, \tag{2), (3}$$

where $\mathbf{u} = (u, v, w)$ is the velocity, P the pressure, T^* the total temperature field, $E = \nu/(\Omega d^2)$ the Ekman number, $\sigma = \nu/k$ the Prandtl number and

$$R = (\alpha g\Delta T d^3)/(\kappa\nu)$$

the Rayleigh number. The quantity T in the momentum equation is defined by

$$T^* = \overline{T^*}+T, \tag{4}$$

which is the deviation of the temperature field from its horizontal mean $\overline{T^*}$. The overbar notation is used throughout to denote the horizontal average, or ensemble average, while the bracket is used for the layer average:

$$\langle f\rangle = \int_0^1 \bar{f}\,dz. \tag{5}$$

The boundary conditions for a free and perfectly conducting top and bottom are such that the vertical component of the velocity and the horizontal shear at top and bottom must vanish:

$$w = 0, \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \quad \text{at} \quad z = 0, 1, \tag{6}$$

$$T^* = \frac{T_0+\Delta T}{\Delta T} \quad \text{at} \quad z = 0, \quad T^* = \frac{T_0}{\Delta T} \quad \text{at} \quad z = 1. \tag{7a, b}$$

By the continuity equation, the vanishing of the horizontal shear implies

$$\partial^2 w/\partial z^2 = 0 \quad \text{at} \quad z = 0, 1. \tag{8}$$

By (4) and (7 *a*, *b*), then, we also have

$$T = 0 \quad \text{at} \quad z = 0, 1. \tag{9}$$

To formulate the variational problem, we now make the following assumptions.

(i) The flow is statistically steady in time and homogeneous in the horizontal averages.

(ii) The horizontal averages of the horizontal velocity components vanish.

(iii) All necessary horizontal averages of the functions describing the flow exist.

Furthermore, we restrict the study to the case where the Prandtl number of the fluid is infinite. This has the advantage of yielding a linear differential constraint from (1):

$$E^{-1}\nabla P = \nabla^2\mathbf{u} + RT\hat{\mathbf{k}} + (2/E)\,\mathbf{u}\times\hat{\mathbf{k}}. \tag{10}$$

Since the nonlinear term $\mathbf{u}.\nabla\mathbf{u}$ is energetically inactive and the flow is assumed statistically steady, the neglected terms have no effect on the energetic balance of the flow. Furthermore, since the Coriolis term $\mathbf{u}\times\hat{\mathbf{k}}$ is also energetically inactive, the linear momentum equation satisfies the same energetic balance as the full momentum equation:

$$R\langle wT\rangle = \langle|\nabla\mathbf{u}|^2\rangle. \tag{11}$$

This equation, usually referred to as the first power integral, is obtained by taking the inner product of $\mathbf{u}$ and the momentum equation, then averaging across the layer. This ensures that the $\sigma = \infty$ assumption will not violate the energetics of the physics. As the rotational constraint does not appear explicitly in (3), the mean temperature

$$-\beta - \overline{wT} \equiv \frac{d\overline{T}^*}{dz} - \overline{wT} = -1 - \langle wT\rangle, \tag{12}$$

and the second-power integral

$$\langle wT\rangle^2 - \langle\overline{wT}^2\rangle + \langle wT\rangle = \langle|\nabla T|^2\rangle, \tag{13}$$

follow in the same manner as in Chan (1971). The Nusselt number Nu, which is the dimensionless heat flux across the layer, is then given by

$$Nu = -\left.\left(\frac{d\overline{T}^*}{dz}\right)\right|_{z=0} = 1 + \langle wT\rangle \tag{14}$$

from (12).

To formulate the variational approach, we now seek to maximize Nu, subject to the integral constraint of (13) and the differential constraint of (10). By the change of scaling

$$\mathbf{u} = \langle wT\rangle^{\frac{1}{2}}R^{\frac{1}{2}}\mathbf{v}, \quad T = \langle wT\rangle^{\frac{1}{2}}R^{-\frac{1}{2}}\theta, \tag{15}$$

where $$\mathbf{v} = (\mathbf{v}.\hat{\mathbf{\imath}})\,\hat{\mathbf{\imath}} + (\mathbf{v}.\hat{\mathbf{\jmath}})\,\hat{\mathbf{\jmath}} + \omega\hat{\mathbf{k}},$$

(10) becomes $$E(\nabla^2\mathbf{v} + \theta\hat{\mathbf{k}}) - \langle wT\rangle^{-\frac{1}{2}}R^{-\frac{1}{2}}\nabla P = 2\hat{\mathbf{k}}\times\mathbf{v}. \tag{16}$$

Next we decompose the solenoidal $\mathbf{v}$ field into a poloidal part $\mathbf{v}_\phi$ and a toroidal part $\mathbf{v}_\psi$:

$$\mathbf{v} = \mathbf{v}_\phi + \mathbf{v}_\psi = -\nabla \times (\nabla \times \phi \hat{\mathbf{k}}) + \hat{\mathbf{k}} \times \nabla \psi;$$

(16) becomes

$$E(\nabla^2 \mathbf{v}_\phi + \nabla^2 \mathbf{v}_\psi + \theta \hat{k}) - \langle wT \rangle^{-\frac{1}{2}} R^{-\frac{1}{2}} \nabla P = 2\hat{\mathbf{k}} \times (-\nabla_1 \phi_z + \nabla_1^2 \phi \hat{\mathbf{k}} + \hat{\mathbf{k}} \times \nabla \psi),$$

where
$$\nabla_1 = \hat{\imath} \frac{\partial}{\partial x} + \hat{\jmath} \frac{\partial}{\partial y}$$

is the horizontal del operator. Taking the $\mathbf{k}$ component of the horizontal curl of the above equation, we have

$$\nabla_1^2 \left(\nabla^2 \psi + \frac{2}{E} \phi_z \right) = 0,$$

or
$$\nabla^2 f + \frac{2}{E} \frac{\partial \omega}{\partial z} = 0, \quad \text{satisfying} \quad f = 0 \quad \text{at} \quad z = 0, 1, \tag{17a, b}$$

where $f = \nabla_1^2 \psi$ is the vertical vorticity component. In the non-rotating case, $E = \infty$, (17a, b) imply that $f = 0$. Here, f is coupled to ω through the rotational constraint and no longer vanishes. Taking the $\hat{\mathbf{k}}$ component of the double curl of (16), we obtain

$$-E(\nabla^4 \omega + \nabla_1^2 \theta) = -2 \frac{\partial}{\partial z} (\hat{\mathbf{k}} . \nabla \times \mathbf{v}) = -2 \frac{\partial}{\partial z} \nabla_1^2 \psi;$$

i.e.
$$\nabla^4 \omega + \nabla_1^2 \theta - \frac{2}{E} \frac{\partial f}{\partial z} = 0. \tag{18}$$

Combining (17a) and (18), we have

$$\nabla^6 \omega + \nabla^2 \nabla_1^2 \theta + T \frac{\partial^2 \omega}{\partial z^2} = 0, \tag{19}$$

where
$$T = \left(\frac{2}{E} \right)^2 = \frac{4 \Omega^2 d^4}{\nu^2} \tag{20}$$

is the Taylor number. With the scaling of (15), the second power integral becomes

$$F = Nu - 1 = \frac{1 - R^{-1} \langle |\nabla \theta|^2 \rangle}{\langle (1 - \overline{\omega \theta})^2 \rangle}, \tag{21}$$

which is obtained by noting that
$$\langle \omega \theta \rangle = 1 \tag{22}$$

and
$$\langle 1 - \overline{\omega \theta^2} \rangle = -\langle \overline{\omega \theta^2} - 2 \overline{\omega \theta} + 1 \rangle = -\langle (1 - \overline{\omega \theta})^2 \rangle.$$

The variational problem of maximizing Nu subject to (10) and (13) is now equivalent to the maximization of

$$G \equiv F + 2\left\langle q' \left(\nabla^4 \omega + \nabla_1^2 \theta - \frac{2}{E} \frac{\partial f}{\partial z} \right) \right\rangle + 2\lambda' \langle \omega \theta - 1 \rangle + 2 \left\langle p' \left(\nabla^2 f + \frac{2}{E} \frac{\partial \omega}{\partial z} \right) \right\rangle, \tag{23}$$

subject to the boundary conditions

$$\theta = \omega = \frac{\partial^2 \omega}{\partial z^2} = \frac{\partial f}{\partial z} = 0 \quad \text{at} \quad z = 0, 1. \tag{24}$$

In (23), $p' = p'(x,y,z)$, $q' = q'(x,y,z)$ and $\lambda' =$ const. are the Lagrange multipliers for (17*a*), (19) and (20), respectively. It is clear that the variational problem of (23) will determine only θ and ω. Yet, once ω and θ are determined, the first power integral, which now assumes the form

$$\langle|\nabla \mathbf{v}|^2\rangle = 1,$$

will automatically be satisfied. Indeed,

$$\begin{aligned}\langle|\nabla \mathbf{v}|^2\rangle &= \langle|\nabla\phi_{xz}+\nabla\psi_y|^2+|\nabla\phi_{yz}-\nabla\psi_x|^2+|\nabla\nabla_1^2\phi|^2\rangle\\ &= -\langle\nabla_1^2\phi\nabla^4\psi\rangle+\langle\psi\nabla^2\nabla_1^2\psi\rangle\\ &= -\left\langle\nabla_1^2\phi\nabla^4\phi+\frac{2}{E}\psi\frac{\partial}{\partial z}\nabla_1^2\phi\right\rangle\\ &= -\left\langle\nabla_1^2\phi\left(\nabla^4\phi-\frac{2}{E}\frac{\partial\psi}{\partial z}\right)\right\rangle = +\langle\omega\theta\rangle = 1,\end{aligned}$$

by using (18) with $\omega = \nabla_1^2\phi$ and $f = \nabla_1^2\psi$.

Using standard procedure, the Euler equations for the variational problem can now be obtained:

$$\nabla^4 q + F(1-\overline{\omega\theta})\,\theta+\lambda\theta-\frac{2}{E}\frac{\partial p}{\partial z} = 0, \tag{25a}$$

$$R^{-1}\nabla^2\theta+\nabla_1^2 q+\lambda\omega+F(1-\overline{\omega\theta})\,\omega = 0, \tag{25b}$$

$$\nabla^4\omega+\nabla_1^2\theta-\frac{2}{E}\frac{\partial f}{\partial z} = 0, \tag{25c}$$

$$\nabla^2 p+\frac{2}{E}\frac{\partial q}{\partial z} = 0, \tag{25d}$$

$$\nabla^2 f+\frac{2}{E}\frac{\partial \omega}{\partial z} = 0, \tag{25e}$$

where

$$q = \frac{q'}{\langle(1-\overline{\omega\theta})^2\rangle},\quad \lambda = \frac{\lambda'}{\langle(1-\overline{\omega\theta})^2\rangle},\quad p = \frac{p'}{\langle(1-\overline{\omega\theta})^2\rangle}.$$

We can now eliminate f, q, p, λ, to arrive at

$$\frac{1}{RF}\nabla^2\left(\nabla^6+T\frac{\partial^2}{\partial z^2}\right)\theta+\left(\nabla^6+T\frac{\partial^2}{\partial z^2}\right)\left[\left(1-\overline{\omega\theta}+\frac{\lambda}{F}\right)\omega\right]-\nabla_1^2\nabla^2\left[\left(1-\overline{\omega\theta}+\frac{\lambda}{F}\right)\theta\right] = 0, \tag{26}$$

$$\left(\nabla^6+T\frac{\partial^2}{\partial z^2}\right)\omega+\nabla_1^2\nabla^2\theta = 0, \tag{27}$$

$$\nabla^2 f+\frac{2}{E}\frac{\partial\omega}{\partial z} = 0, \tag{28}$$

satisfying the boundary conditions,

$$\omega = \frac{\partial^2\omega}{\partial z^2} = \frac{\partial^4\omega}{\partial z^2} = \theta = \frac{\partial^2\theta}{\partial z^2} = \frac{\partial^4\theta}{\partial z^2} = \frac{\partial^6\theta}{\partial z^2} = \frac{\partial f}{\partial z} = 0 \quad \text{at} \quad f = 0, 1. \tag{29}$$

Next we shall show that $\lambda \sim O(1)$ regardless of what R and T are. To see this, we multiply (25a) by ω, (25b) by θ, (25c) by q, then take the layer average of these equations; we have

$$\langle \omega \nabla^4 q\rangle - F\langle (1-\overline{\omega\theta})^2\rangle + \lambda - \frac{2}{E}\left\langle \omega \frac{\partial p}{\partial z}\right\rangle = 0,$$

$$-R^{-1}\langle |\nabla\theta|^2\rangle - F\langle (1-\overline{\omega\theta})^2\rangle + \lambda + \langle \theta \nabla_1^2 q\rangle = 0,$$

$$\langle q\nabla^4\omega\rangle + \langle q\nabla_1^2\theta\rangle - \frac{2}{E}\left\langle q\frac{\partial f}{\partial z}\right\rangle = 0.$$

Subtracting the last equation from the sum of the first two, using the fact that

$$\langle \omega\nabla^4 q\rangle = \langle q\nabla^4\omega\rangle, \quad \langle\theta\nabla_1^2 q\rangle = \langle q\nabla_1^2\theta\rangle,$$

$$2\lambda - 2F\langle (1-\overline{\omega\theta})^2\rangle - R^{-1}\langle|\nabla\theta|^2\rangle + \frac{2}{E}\left\langle q\frac{\partial f}{\partial z} - \omega\frac{\partial p}{\partial z}\right\rangle = 0.$$

Making use of the boundary conditions on f, ω, p and q, using (25d, e),

$$\frac{2}{E}\left\langle q\frac{\partial f}{\partial z} - \omega\frac{\partial p}{\partial z}\right\rangle = -\frac{2}{E}\left\langle f\frac{\partial q}{\partial z} - p\frac{\partial \omega}{\partial z}\right\rangle = -\langle f\nabla^2 p - p\nabla^2 f\rangle = 0;$$

i.e. $$2\lambda = 2 - R^{-1}\langle|\nabla\theta|^2\rangle \quad\text{or}\quad \tfrac{1}{2} \leqslant \lambda \equiv \tfrac{1}{2} + \tfrac{1}{2}(1 - R^{-1}\langle|\nabla\theta|^2\rangle) \leqslant 1, \tag{30}$$

since F must be positive definite, so that

$$0 \leqslant 1 - R^{-1}\langle|\nabla\theta|^2\rangle \leqslant 1.$$

The fact that $\lambda \sim O(1)$ will later on be used in the boundary-layer analysis.

To maximize F, we must solve the nonlinear equations (27)–(28). As in the case where $\Omega = 0$, the nonlinearity is of a form such that they allow solutions whose horizontal dependence is separable, i.e.

$$\omega = \sum_{n=1}^{N} \omega_n(z)\, h_n(x,y), \tag{31}$$

etc., with $$\nabla_1^2 h = -\alpha_n^2 h_n \quad\text{and}\quad \langle h_n h_m\rangle = \delta_{nm}, \tag{32}$$

where δ_{nm} is the Kronecker delta.

By considering these separable solutions, (27)–(28) now become a nonlinear ordinary differential system, for $n = 1$ to N:

$$\frac{1}{RF_N}\left(\frac{d^2}{dz^2} - \alpha_n^2\right)\left[\left(\frac{d^2}{dz^2} - \alpha_n\right)^3 + T\frac{d^2}{dz^2}\right]\theta_n + \left[\left(\frac{d^2}{dz^2} - \alpha_n^2\right)^3 + T\frac{d^2}{dz^2}\right]$$
$$\times\left[\left(1 - \sum_1^N \omega_n\theta_n + \frac{\lambda}{F_N}\right)\omega_n\right] + \alpha_n^2\left(\frac{d^2}{dz^2} - \alpha_n^2\right)\left[\left(1 - \sum_1^N \omega_n\theta_n + \frac{\lambda}{F_N}\right)\theta_n\right] = 0, \tag{33}$$

$$\left[\left(\frac{d^2}{dz^2} - \alpha_n^2\right)^3 + T\frac{d^2}{dz^2}\right]\omega_n - \alpha_n^2\left(\frac{d^2}{dz^2} - \alpha_n^2\right)\theta_n = 0. \tag{34}$$

As we are still unable to solve the system for arbitrary values for R and T, we shall restrict ourselves in the following to boundary-layer solutions as $F \to \infty$ with $R \to \infty$ and $T \to \infty$.

Before we do that, however, it is worth noting that another asymptotic solution corresponding to $F \ll 1$ can also be studied. If $F \gg 1$ is to be interpreted as the turbulent regime where the convective heat flux is large, $F \ll 1$ must then be related to the onset of convective motion. In fact, expressing both ω and θ as power series of F immediately yields the same equation used in linear stability, with higher-order terms consistent with finite-amplitude convection results (Chan 1972). This fact then enables us to use results from the linear stability theory within the maximum principle.

3. The single-α boundary-layer solution

While it is possible to obtain the multi-α solution directly by a procedure similar to that used in the non-rotating case, we shall first study the single-α structure in detail (i.e. $N = 1$ in (31)). This will then provide much insight into the multi-α structure to be discussed in §4.

Consider a conceptual experiment where both R and T are large enough that the flow is in a turbulent regime (i.e. $F \gg 1$). Suppose R is now fixed, and T is increased gradually. There must come a point at which no convection is possible, since the linear theory predicts that convection cannot be maintained unless $R \geqslant O(T^{\frac{2}{3}})$. Mathematically it suggests the necessity of considering two different asymptotic functional dependences for F, depending on the relation between R and T. In the non-rotating case, the condition $F \gg \alpha \gg 1$ is always maintained, where the wavenumber α determines the length scale of the eddies, so to speak. In the rotating case, therefore, we seek the two different ranges of F as $F \gg \alpha \gg 1$ and $\alpha \gg F \gg 1$ for moderately and extremely large values of T (in relation to R). This observation is in fact one of the basic differences between the study of rotating and non-rotating cases. With this in mind, we now proceed to the formal asymptotic solution.

For the single-α case, the Euler–Lagrange equation is easily obtained from (33)–(34) by setting $N = 1$. As in the non-rotating case, it is assumed that $F_1 \gg 1$ and $\alpha_1 \gg 1$, where the subscript on F indicates the single-α constraint. To obtain the boundary-layer solution, it is found that three different regions are needed: the interior, the transition layer and finally the boundary layer. What their thickness is will become clear in the following analysis.

In the interior region where $(d/dz) \sim O(1)$, the balance of (33) and (34) can be obtained easily by keeping z fixed and letting R, T, α and F go to infinity, taking

$$\alpha_1^4 \ll T \ll \alpha_1^6. \tag{35}$$

This then results in

$$-\alpha_1^6\omega_1 + \alpha_1^4\theta_1 = 0 \tag{36}$$

and

$$\frac{\alpha_1^8}{RF_1}\theta_1 - \alpha_1^6(1-\omega_1\theta_1)\,\omega_1 + \alpha_1^4(1-\omega_1\theta_1)\,\theta_1 = 0, \tag{37}$$

which suggest that we take

$$\omega_1 = \alpha_1^{-1}\tilde{\omega}_1, \quad \theta_1 = \alpha_1\tilde{\theta}_1, \tag{38}$$

with both $\tilde{\omega}_1$ and $\tilde{\theta}_1$ to be $O(1)$. Using (30), F can now be expressed as

$$F = \frac{2\lambda - 1}{\langle(1-\overline{\omega\theta})^2\rangle}, \tag{39}$$

where the numerator is of $O(1)$. $F \gg 1$ then clearly requires that

$$1-\overline{\omega\theta} = 1-\omega_1\theta_1 \simeq 0$$

holds in a large part of the fluid. This can be achieved by assuming that

$$\alpha_1^4/RF \ll 1 \tag{40}$$

and substituting (38) into (37). Thus, we have, in the interior,

$$\tilde{\omega}_1 = \tilde{\theta}_1 = 1. \tag{41}$$

Presumably, then, ω_1 and θ_1 (which vanish at the boundaries) must build up rather rapidly in a thin layer towards their interior value of α_1^{-1} and α_1, respectively. As in the non-rotating case, such a matching demands the existence of a transition layer.

Assume now that the transition layer is of order $\epsilon_1 \ll 1$. We write

$$z-z_0 = \epsilon_1\xi_1 \quad \text{for} \quad z_0 = 0 \quad \text{or} \quad 1, \tag{42}$$

and
$$\omega_1 = \alpha_1^{-1}\mathring{\omega}_1(\xi_1), \quad \theta_1 = \alpha_1\mathring{\theta}_1(\xi), \tag{43}$$

in view of (38). Equation (34) now becomes

$$[(\epsilon_1^{-2}D^2-\alpha_1^2)^3+(T/\epsilon_1^2)D^2]\mathring{\omega}_1-\alpha_1^4(\epsilon_1^{-2}D^2-\alpha_1^2)\mathring{\theta}_1' = 0, \tag{44}$$

where $D = d/d\xi_1$. Physically, this is the layer where the effect of rotation begins to take effect. This requires mathematically that the balance

$$\alpha_1^{-2} \ll \epsilon_1^2 = T/\alpha_1^6 \ll 1 \tag{45}$$

be taken, which in turn poses a limit for T beyond which either convection cannot be maintained, or else the rotation will be too weak to be felt:

$$\alpha_1^4 \ll T \ll \alpha_1^6. \tag{46}$$

Holding ξ_1 fixed, and letting R, T and F go to infinity, (33) and (34) become

$$(D^2-1)\mathring{\omega}_1+\mathring{\theta}_1 = 0 \quad \text{and} \quad 1-\mathring{\omega}_1\mathring{\theta}_1 = 0. \tag{47), (48}$$

In this layer, only one of the boundary conditions for ω_1 can be satisfied. Together with asymptotic matching, $\mathring{\omega}_1$ must then satisfy the conditions

$$\mathring{\omega}_1(0) = 0 \quad \text{and} \quad \mathring{\omega}_1(\xi_1\to\infty) = 1. \tag{49}$$

While the existence of such a solution can easily be established, it is only important that its form as $\xi_1\to 0$ be obtained as

$$\theta_1^{-1} = \omega_1 \sim \alpha_1^{-1}\xi_1\log^{\frac{1}{2}}\xi_1^{-2}+\dots. \tag{50}$$

Finally, for the boundary-layer balance, let

$$z-z_0 = \frac{\alpha_1}{T^{\frac{1}{2}}}g_1\zeta_1 = \frac{T^{\frac{1}{2}}}{\alpha_1^3}\xi_1. \tag{51}$$

Anticipating the matching condition as $\zeta_1\to\infty$, (50) and (51) suggest that

$$\omega_1 = \frac{\alpha_1^3 g_1}{T}\left(\log\frac{T^2}{\alpha_1^8 g_1^2}\right)^{\frac{1}{2}}\hat{\omega}_1(\zeta_1), \quad \theta_1 = \frac{T}{\alpha_1^3 g_1}\left(\log\frac{T^2}{\alpha_1^8 g_1^2}\right)^{-\frac{1}{2}}\hat{\theta}_1(\zeta_1), \tag{52}$$

with
$$\hat{\omega}_1(\zeta_1)\to\zeta_1 \quad \text{and} \quad \hat{\theta}_1\to\zeta_1^{-1} \quad \text{as} \quad \zeta_1\to\infty. \tag{53}$$

It is now important to consider the two different cases for $F_1 \sim O(T^{\frac{1}{2}}/(\alpha_1 g_1))$,

$$\text{(i) } F_1 \gg \alpha_1 \gg 1 \quad \text{or} \quad \text{(ii) } \alpha_1 \gg F_1 \gg 1. \tag{54}$$

For (i), the inner limit of (34) is now

$$\left[\frac{T^3}{\alpha_1^6 g_1^6} D^6 + \frac{T^2}{\alpha_1^2 g_1^2} D^2\right] \hat{\omega}_1 = T^3 \left[\alpha_1^6 g_1^4 \log \frac{T^2}{\alpha_1^8 g_1^2}\right]^{-1} D^2 \hat{\theta}_1,$$

where $D = d/d\zeta_1$. Anticipating that $F_1 \gg T^{\frac{1}{4}}$ and that $g_1 \ll 1$, it is further reduced to

$$D^6 \hat{\omega}_1 = 0. \tag{55}$$

The boundary and the matching conditions then give

$$\hat{\omega}_1 = \zeta_1. \tag{56}$$

Similarly, (33) then becomes, under the same assumptions,

$$T^3 \left[R F_1 \alpha_1^8 g_1^4 \log \frac{T^2}{\alpha_1^8 g_1^2}\right]^{-1} D^8 \hat{\theta}_1 + D^6[(1-\hat{\omega}_1 \hat{\theta}_1)\, \hat{\omega}_1] = 0. \tag{57}$$

The appropriate balance is now taken to be

$$T^3 \left[R F_1 \alpha_1^8 g_1^4 \log \frac{T^2}{\alpha_1^8 g_1^2}\right]^{-1} = 1. \tag{58}$$

Thus, $\hat{\theta}_1$ must satisfy

$$D^2 \hat{\theta}_1 + (1 - \zeta_1 \hat{\theta}_1)\, \zeta_1 = 0 \tag{59}$$

with

$$\hat{\theta}_1(0) = 0, \quad \hat{\theta}_1(\zeta_1 \to \infty) = \zeta_1^{-1}, \tag{60}$$

i.e.

$$\hat{\theta}_1 = \tfrac{1}{2}\zeta_1 \int_0^1 \exp\{-\tfrac{1}{2}\zeta^2 t\}\,(1-t^2)^{\frac{1}{2}}\, dt. \tag{61}$$

F_1 can now be computed as a function of α_1, which will then be chosen so as to yield a maximum F_1.

Using the separable solution, with $N = 1$,

$$\langle |\nabla \theta|^2 \rangle = \sum_1^N \left\langle \left(\frac{d\theta_n}{dz}\right)^2 + \alpha_n^2 \theta_n^2 \right\rangle = \alpha_1^4 + 2\beta T^{\frac{1}{2}} \left[\alpha_1^7 g_1^3 \log \frac{T^2}{\alpha_1^8 g_1^2}\right]^{-1}$$

and

$$F_1 = \frac{T^{\frac{1}{2}}}{\alpha_1 g_1} 2\sigma^{-1} \left[1 - R^{-1}\left(\alpha_1^4 + 2\beta T^{\frac{5}{2}} \left[\alpha_1^7 g_1^3 \log \frac{T^2}{\alpha_1^8 g_1^2}\right]^{-1}\right)\right], \tag{62}$$

where

$$\sigma = \int_0^\infty (1 - \zeta_1 \hat{\theta}_1)^2\, d\zeta_1 \quad \text{and} \quad \beta = \int_0^\infty \left(\frac{d\hat{\theta}_1}{d\zeta_1}\right)^2 d\zeta_1,$$

and the factor 2 in (62) accounts for the two boundary layers, at $z = 0$ and 1. Using (58) and defining

$$I_+ = \sigma + \beta, \quad F_1 = 2 I_+^{-1} \frac{T^{\frac{1}{2}}}{\alpha_1 g_1}\left(1 - \frac{\alpha_1^4}{R}\right). \tag{63), (64}$$

Setting $\partial F_1/\partial \alpha_1 = 0$, and using (58) again, one obtains

$$\frac{\partial g_1}{\partial \alpha_1} = -\frac{2 g_1}{\alpha_1} \quad \text{and} \quad \alpha_1 = (\tfrac{1}{5}R)^{\frac{1}{4}}. \tag{65), (66}$$

Straightforward calculation then shows

$$g_1 \simeq (3I_+)^{\frac{1}{3}} \left(\frac{5}{R}\right)^{\frac{11}{12}} T^{\frac{5}{6}} \left(\log \frac{T^2}{R}\right)^{-\frac{1}{3}} \tag{67}$$

and

$$F_1 = \frac{2}{5^{\frac{5}{3}}} 3^{-\frac{1}{3}} (I_+)^{-\frac{4}{3}} \frac{R^{\frac{2}{3}}}{T^{\frac{1}{3}}} \left(\log \frac{T^2}{R}\right)^{\frac{1}{3}}. \tag{68}$$

The various assumptions on ordering now indicate that (68) is valid as long as

$$O(R) \ll T \ll O(R^{\frac{11}{10}}),$$

owing primarily to the fact that $g_1 \ll 1$ is assumed.

For case (ii) of (54), the proper balance for the boundary layer must be different because here, $\nabla^2 \sim -\alpha_1^2$ instead. Keeping this in mind, (34) readily becomes

$$\frac{T^2}{\alpha_1^2 g_1^2} D^2 \hat{\omega}_1 + \alpha_1^4 T^2 \left[\alpha_1^6 g_1^2 \log \frac{T^2}{\alpha_1^8 g_1^2}\right]^{-1} \hat{\theta}_1 = 0, \quad \text{i.e.} \quad D^2 \hat{\omega}_1 = 0. \tag{69}$$

Likewise, (33) becomes

$$-\frac{1}{RF_1} \alpha_1^2 T^2 \left[\alpha_1^6 g_1^2 \log \frac{T^2}{\alpha_1^8 g_1^2}\right]^{-1} D^2 \hat{\theta}_1 + D^2 (1 - \hat{\omega}_1 \hat{\theta}_1) \hat{\omega}_1 = 0. \tag{70}$$

The proper balance is therefore

$$T^2 \left[RF_1 \alpha_1^4 g_1^2 \log \frac{T^2}{\alpha_1^8 g_1^2}\right]^{-1} = 1. \tag{71}$$

Integrating (69) and (70) with the boundary and matching conditions now yields

$$\hat{\omega}_1 = \zeta_1 \quad \text{and} \quad \hat{\theta}_1 = \zeta_1 / (1 + \zeta_1^2). \tag{72}$$

A straightforward computation, similar to those above, then gives

$$F_1 = \frac{T^{\frac{1}{2}}}{\alpha_1 g_1} 2 I_-^{-1} \left(1 - \frac{\alpha_1^4}{R}\right), \tag{73}$$

where

$$I_- = \int_0^\infty \left(1 - \frac{\zeta_1^2}{1 + \zeta_1^2}\right)^2 d\zeta_1 = \tfrac{1}{4}\pi. \tag{74}$$

In (73), the term due to $\langle (d\theta/dz)^2 \rangle$ (i.e. the term that corresponds to the last term of the right-hand side of (62)) is negligible compared with F_1, in view of (71). Maximizing F_1 with respect to α_1 now yields

$$g_1 \simeq \frac{5^{\frac{7}{4}} I_-}{2} \frac{T^{\frac{3}{2}}}{R^{\frac{7}{4}}} \left(\log \frac{R^{\frac{3}{2}}}{T}\right)^{-1}, \quad \alpha_1 = (\tfrac{1}{5} R)^{\frac{1}{4}}, \tag{75), (76}$$

and

$$F_1 = \frac{2^2}{5^{\frac{5}{2}}} I_-^{-2} \frac{R^{\frac{3}{2}}}{T} \log \frac{R^{\frac{3}{2}}}{T}, \tag{77}$$

valid for

$$O(R^{\frac{5}{4}} \log R^{\frac{3}{2}}/T) \leqslant T \leqslant O(R^{\frac{3}{2}}), \tag{78}$$

the restriction on the lower end being required by the assumption that $F_1 \leqslant \alpha_1$ here.

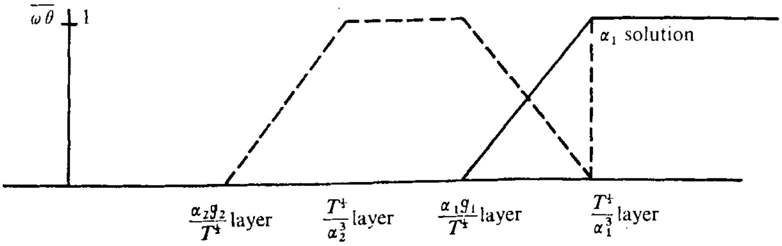

FIGURE 2. Schematic structure of the multi-α solution.

This form for the convective heat flux is particularly interesting when compared with the linear theory. According to linear stability analysis, the onset of convection for large T occurs when $R = O(T^{\frac{2}{3}})$. One might then expect that, for $R \sim O(T^{\frac{2}{3}})$, F will then be $O(1)$. Equation (77), indeed, predicts the same qualitative behaviour when the result is extrapolated to $T \sim O(R^{\frac{3}{2}})$, though it may not be quantitatively correct. As we shall see, the same qualitative feature is to persist for the multi-α structure as well.

4. The multi-α boundary-layer solution

As in the non-rotating case, the purpose of seeking a multi-α solution is to increase F by an order of magnitude. The form of F as in (39) suggests that this can be achieved by making $1-\overline{\omega\theta} \simeq 0$ for as large a range as possible. F^{-1} will then be of the same order of magnitude as the boundary layer in which $1-\overline{\omega\theta}$ is not asymptotic to zero. Suppose that, for certain values of R and T, the single-α solution is able to maintain a certain boundary layer of thickness, say, $\delta_1 \ll 1$. If a second mode α_2 can be introduced, so that $1-\overline{\omega\theta}$ is now asymptotic to zero in the δ_1 layer, and differs from zero only in a layer of thickness $\delta_2 \ll \delta_1$, the value for F corresponding to such a solution will thus be increased from its single-α value by an order of magnitude. Higher modes may then be introduced similarly. This essentially corresponds to the physical stituation of the breaking up of the flow into smaller eddies. Schematically, the introduction of higher modes can be represented by the structure of figure 2. Such a structure is now complicated by the presence of the second parameter T. As was explained previously, it must be expected that, as T increases while R is held fixed, F will eventually have to decrease towards zero (i.e. for each additional mode, two ranges for F must again be considered).

Consider the process of maximizing F by adding one additional mode at a time, with the notation F_n to denote the value of F as a result of the nth mode being added. F_N is of course just F when there are altogether N modes. As shown in §3, when $N = 1$, there are two different cases, ((i) $F_1 \gg \alpha_1 \gg 1$ and (ii) $\alpha_1 \gg F_1 \gg 1$), corresponding to two different regions in the R, T space (i.e. for a fixed R, assumed to be large, F_1 will go from the first range to the second one as T is increased beyond a certain order of R). To construct a solution having two wavenumbers, then, the value F_2, for the same value of R and T, must be an order of magnitude larger than F_1. In range (i) above, we thus seek a solution

n (mode number)	k (section number) 0	1	2	…	$N-1$	N
1	$F_1 > \alpha_1$	$\alpha_1 > F_1 \gg 1$				
2	$F_2 > \alpha_2$	$F_2 > \alpha_2$	$\alpha_2 \gg F_2 \gg 1$			
⋮						
N	$F_N > \alpha_N$	$F_N > \alpha_N$			$F_N > \alpha_N$	$\alpha_N > F_N \gg 1$

TABLE 1. Recursive relationship between F_n and α_n

$F_2 \gg \alpha_2 \gg \alpha_1$ while $F_2 \gg F_1$. As T is increased to values in range (ii) above, we continue to seek a solution having two horizontal wavenumbers such that $F_2 \gg \alpha_2 \gg \alpha_1$ and $F_2 \gg F_1$ remain true. This cannot be maintained for indefinitely increasing values of T however. For T large enough (keeping R fixed), convection is expected to stop. Mathematically, it must correspond to a different functional dependence of F_2 on R and T, so that $\alpha_2 \gg F_2 \gg F_1$ now holds instead (i.e. for a 2-α solution, there must be three different regions (i)–(iii) in the R, T space such that, in (i) $F_2 \gg \alpha_2$ and $F_1 \gg \alpha_1$, in (ii) $F_2 \gg \alpha_2$ and $\alpha_1 \gg F_1$ with $F_2 \gg F_1$, and in (iii), where T is extremely large, $\alpha_2 \gg F_2$, $\alpha_1 \gg F_1$, though $F_2 \gg F_1$ continues to hold). Similarly for each additional mode. It is thus easy to see that, for a solution having N horizontal modes, there must be $N+1$ different regions in the parameter space, in each of which the relation between F_n and α_n changes one at a time, in the manner described above. These different regions may be obtained by keeping R fixed as T increases. Each time, as T is increased beyond a certain order of R, one of the F_n, α_n pair will change their relative order. Denoting each of these R, T regions by a section number k, we summarize the above behaviour in table 1 (e.g. in section $N-1$, i.e. for T lying in between two R-orders, we shall have $\alpha_1 \gg F_1$, $\alpha_2 \gg F_2 \ldots, \alpha_{N-1} \gg F_{N-1}$, but $F_N \gg \alpha_N$, whereas, in the Nth section, we shall have instead $\alpha_i \gg F_i$ for all $i = 1$ to N).

While structures of the type shown in table 1 assume that in any section it is possible to seek a solution having more and more horizontal modes, it turns out that such an indefinite increase in the value of N is not always possible; but this is not obvious *a priori*. For the purpose of a systematic study, we should nonetheless assume the possibility of such an indefinite increase in N, then proceed to determine the condition under which it is indeed possible.

In the following analysis, we shall denote all quantities by a double subscript (n, k) where n represents the horizontal mode and k indicates the validity of the solution in certain range of R, T relation, yet to be determined.

By analogy with the single-α structure, we again need three different layers for each mode. Take, for example, the $(n+1)$th mode. There will be an interior region which is the boundary-layer region of the nth mode, a transitional layer of $O(T^{\frac{1}{2}}/\alpha_{n+1}^3)$, and finally a boundary layer of $O(\alpha_{n+1} g_{n+1}/T^{\frac{1}{2}})$. We shall also use the notation F_n for $n < N$ with F_n^{-1} designating the thickness of the nth boundary layer (i.e. the boundary-layer thickness of $1-\overline{\omega\theta}$ if the nth mode were the highest-order mode). Of course, $F = F_N$.

In the various layers, we now write, for $n = 1$ to N,

$$\omega_{n,k}(z) = \begin{Bmatrix} A_{n,k}\hat{\omega}_{n,k}(\zeta_{n,k}) \\ B_{n,k}\mathring{\omega}_{n,k}(\xi_{n,k}) \\ C_{n,k}\tilde{\omega}_{n,k}(\zeta_{n-1,k}) \end{Bmatrix}, \quad \theta_{n,k}(z) = \begin{Bmatrix} A_{n,k}^{-1}\hat{\theta}_{n,k}(\zeta_{n,k}) & & z \sim O\left(\dfrac{\alpha_{n,k}g_{n,k}}{T^{\frac{1}{2}}}\right), \\ B_{n,k}^{-1}\mathring{\theta}_{n,k}(\xi_{n,k}) & \text{for} & z \sim O\left(\dfrac{T^{\frac{1}{2}}}{\alpha_{n,k}^{3}}\right), \\ C_{n,k}^{-1}\tilde{\theta}_{n,k}(\zeta_{n-1,k}) & & z \sim O\left(\dfrac{\alpha_{n-1,k}g_{n-1,k}}{T^{\frac{1}{2}}}\right). \end{Bmatrix} \tag{79}$$

The relative ordering for ω and θ is chosen as above by the requirement that $\overline{\omega\theta} \sim O(1)$ throughout the boundary layer and beyond. To proceed, assumptions must now be made about the ordering of the $\alpha_{n,k}$'s. The structure of the solution assumes that higher modes have a shorter length scale, so it is natural to assume that

$$\alpha_{n+1,k} \gg \alpha_{n,k} \quad \text{for all} \quad n, k. \tag{80}$$

By analogy with the single-α solution, the balance of (34) in the interior of the nth mode (i.e. the $\zeta_{n-1,k}$ layer) necessitates the additional requirement that

$$\alpha_{n,k}^{6} \gg T^2/(\alpha_{n-1,k}^{2}g_{n-1,k}^{2}). \tag{81}$$

Again, the consideration that in the $\zeta_{n-1,k}$ layer

$$1-\overline{\omega\theta} \sim 1-\tilde{\omega}_{n,k}\tilde{\theta}_{n,k}-\hat{\omega}_{n-1,k}\hat{\theta}_{n-1,k} = 0 \tag{82}$$

also requires, from (33),

$$\alpha_{n,k}^{4}/(RF_N) \ll 1 \quad (1 \leqslant n \leqslant N). \tag{83}$$

In any event, in this layer, either by arguments such as those above, or by making formal balance of (33) and (34), with appropriate assumptions such as (81) and (82), one readily obtains

$$A_{n,k} = B_{n,k} = \alpha_{n,k}^{-1} \quad (1 \leqslant n \leqslant N). \tag{84}$$

The solution in the $\zeta_{n-1,k}$ layer is then

$$\tilde{\omega}_{n,k} = \tilde{\theta}_{n,k}, \tag{85}$$

in addition to (82). As for the transitional layer, reasoning similar to that used in obtaining the single-α solution, gives

$$\mathring{\omega}_{n,k}(D^2-1)\,\mathring{\omega}_{n,k}+1 = 0, \quad D = d/d\xi_{n,k}, \tag{86}$$

with
$$\mathring{\omega}_{n,k}(0) = 0 \quad \text{and} \quad \mathring{\omega}_{n,k}(\xi_{n,k}\to\infty) = 1, \tag{86}$$

while
$$\mathring{\theta}_{n,k}(\xi_{n,k}) = 1/\mathring{\omega}_{n,k}(\xi_{n,k}). \tag{87}$$

These are clearly the same as $\hat{\omega}_1(\xi_1)$ functionally, so the matching condition for $\hat{\omega}_{n,k}$ now likewise requires that

$$C_{n,k} = \frac{\alpha_{n,k}^3 g_{n,k}}{T}\left(\log\frac{T^2}{\alpha_{n,k}^8 g_{n,k}^2}\right)^{\frac{1}{2}}. \tag{88}$$

Using these values, the formal balance of (34), in a manner like that of the single-α case, now gives

$$D^2\hat{\omega}_{n,k} = 0 \quad \text{with} \quad \hat{\omega}_{n,k}(0) = 0 \quad \text{and} \quad \hat{\omega}_{n,k}(\zeta_{n,k}\to\infty) = \zeta_{n,k},$$

or simply

$$\hat{\omega}_{n,k} = \zeta_{n,k}. \tag{89}$$

Finally, the equations for $\hat{\theta}_{n,k}$ can now be obtained either by formally taking the inner limit of (33) and (34), or by an appropriate maximizing consideration. In any event, direct computation gives

$$\langle|\nabla\theta|^2\rangle = \sum_{n=1}^{N}\left\{2T^{\frac{5}{4}}\left[\alpha_{n,k}^7 g_{n,k}^3\log\frac{T^2}{\alpha_{n,k}^8 g_{n,k}^2}\right]^{-1}\left\langle\left(\frac{d\hat{\theta}_{n,k}}{d\zeta_{n,k}}\right)^2\right\rangle + 2\alpha_{n,k}^4\frac{\alpha_{n-1,k}g_{n-1,k}}{T^{\frac{1}{2}}}\langle\hat{\theta}_{n,k}^2\rangle\right\}, \tag{90}$$

where the notation $\langle\ \rangle$ indicates an integration from 0 to infinity against the appropriate argument of the function for all $n \geqslant 1$, and the convention that $\frac{\alpha_{0,k}\,g_{0,k}}{T^{\frac{1}{2}}}\langle\hat{\theta}_{1,k}^2\rangle = \int_0^{\frac{1}{2}} dz = \frac{1}{2}$ is used. For F_N to be a maximum, (21) now suggests that the expression in (90) must be of order R. In the non-rotating case, where $F_n > \alpha_n$ is always true, this is accomplished by making every term in (90) of order R, yielding therefore $2N$ equations for the $2N$ quantities $\alpha_{n,k}$ and $g_{n,k}$. With the rotational constraint, however, $F_{n,k} \gg \alpha_{n,k}$ is not always possible, as illustrated in table 1. The single-α solution shows that the effect of $F_{n,k} \ll \alpha_{n,k}$ is to make the contribution from the nth mode to the term $\langle(d\theta/dz)^2\rangle$ negligible. A reference to table 1 now clearly indicates that (90) will be reduced, for asymptotic purposes, to

$$\langle|\nabla\theta|^2\rangle = 2\sum_{n=1}^{N}\alpha_{n,k}^4\frac{\alpha_{n-1,k}g_{n-1,k}}{T^{\frac{1}{2}}}\langle\hat{\theta}_{n,k}^2\rangle + 2\sum_{k+1}^{N}T^{\frac{5}{4}}\left[\alpha_{n,k}^7 g_{n,k}^3\log\frac{T^2}{\alpha_{n,k}^8 g_{n,k}^2}\right]^{-1}\left\langle\left(\frac{d\hat{\theta}_{n,k}}{d\zeta_{n,k}}\right)^2\right\rangle, \tag{91}$$

by dropping the terms for $n \leqslant k$. The requirement that these terms be of order R will now provide only $2N-k$ equations:

$$\alpha_{n,k}^4\frac{\alpha_{n-1,k}g_{n-1,k}}{\sqrt{T}} \sim R \quad (1 \leqslant n \leqslant N), \tag{92}$$

and

$$T^{\frac{5}{4}}\left[\alpha_{n,k}^7 g_{n,k}^3\log\frac{T^2}{\alpha_{n,k}^8 g_{n,k}^2}\right]^{-1} \sim R \quad (k+1 \leqslant n \leqslant N). \tag{93}$$

Another k constraints will now be obtained by taking

$$T^{\frac{5}{4}}\left[\alpha_{n,k}^3 g_{n,k}\log\frac{T^2}{\alpha_{n,k}^8 g_{n,k}^2}\right]^{-1} \sim R \quad (1 \leqslant n \leqslant k) \tag{94}$$

by analogy with (71), with the consideration that $F_{n,k} \sim T^{\frac{1}{2}}/(\alpha_{n,k} g_{n,k})$. In fact, the formal balance of (33) for $\tilde{\theta}_{n+1,k}$ and $\hat{\theta}_{n,k}$ for $1 \leqslant n \leqslant k$ requires precisely

$$\frac{1}{RF_N} \alpha^4_{n+1,k} \sim \frac{\alpha^2_{n,k}}{RF_N} T^2 \left[\alpha^6_{n,k} g^2_{n,k} \log \frac{T^2}{\alpha^8_{n,k} g^2_{n,k}} \right]^{-1}, \tag{95}$$

which together with (92) imply (94). Here, we use the convention

$$\frac{\alpha_{0,k} g_{0,k}}{T^{\frac{1}{2}}} = \tfrac{1}{2} \quad \text{and} \quad (\;)_{j,j+s} = (\;)_{j,j} \quad \text{for} \quad s \geqslant 0, \tag{96}$$

in accordance to the relations of table 1. Keeping in mind that k is considered fixed, (92) to (94) must now be divided into two systems: for $1 \leqslant n \leqslant k$ and $k+1 \leqslant n \leqslant N$. In solving the second system, the aim is to express all the $\alpha_{n,k}$'s, etc., in terms of the $\alpha_{k,k}$'s etc. As for the first system, (96) indicates that it must be solved recursively for k instead, until everything is in terms of $\alpha_{n,n}$, etc.

In any event, for $n > k$, writing (93) as

$$\frac{\alpha_{n-1,k} g_{n-1,k}}{T^{\frac{1}{2}}} \sim \frac{T^{\frac{1}{3}}}{R^{\frac{1}{3}} \log^{\frac{1}{3}}(n-1,k)} \alpha^{-\frac{4}{3}}_{n-1,k},$$

and, using (92), one obtains recursively

$$\alpha_{n,k} \sim \left(\frac{R^{\frac{1}{3}}}{T^{\frac{1}{12}}} \right) \alpha^{\frac{1}{3}}_{n-1,k} \log^{\frac{1}{12}}(n-1,k),$$

where
$$\log(n,k) = \log \frac{T^2}{\alpha^8_{n,k} g^2_{n,k}}.$$

Reducing n downward to k now yields, for $n > k$,

$$\alpha_{n,k} = b_{n,k} \left(\frac{R^{\frac{1}{3}}}{T^{\frac{1}{12}}} \right)^{\frac{3}{2}(1-3^{1+k-n})} (\alpha_{k+1,k})^{3^{1+k-n}} \prod_{j=1}^{n-k-1} \log^{\frac{1}{4}3^{-j}}(n-j,k),$$

where $b_{n,k}$ is a constant. Using (92) for $n = k+1$, and (94) for $n = k$, now gives

$$\alpha_{k+1,k} \sim (R^2/T)^{\frac{1}{4}} \alpha^{\frac{1}{2}}_{k,k} \log^{\frac{1}{4}}(k,k),$$

so that, for $n > k$,

$$\alpha_{n,k} = b_{n,k} \frac{R^{\frac{1}{2}}}{T^{\frac{1}{8}(1+3^{1+k-n})}} \alpha^{\frac{1}{2}3^{1+k-n}}_{k,k} \log^{\frac{1}{4}3^{1+k-n}}(k,k) \prod_{j=1}^{n-k-1} \log^{\frac{1}{4}3^{-j}}(n-j,k). \tag{97}$$

Using (92), (94), and (96) to solve for k recursively for $1 \leqslant n \leqslant k$ now yields

$$\alpha^4_{n,n} \sim \frac{R^2}{T} \alpha^2_{n-1,n-1} \log(n-1,n-1), \quad \frac{\alpha_{n,n} g_{n,n}}{T^{\frac{1}{2}}} \sim \frac{T}{R \log(n,n)} \alpha^{-2}_{n,n},$$

or
$$\alpha_{n,n} = b_{n,n} \frac{R^{\frac{1}{2}(2-3.2^{-n})}}{T^{\frac{1}{2}(1-2.2^{-n})}} \sum_{j=1}^{n-1} \log^{2(j+1)^{-1}}(n-j,n-j), \tag{98}$$

where we have used the known fact that $R_{1,0} = R_{1,1} \sim R^{\frac{1}{4}}$. Upon using (97) and (98) jointly, we finally obtain

$$\alpha_{n,k} = b_{n,k} \frac{R^{\frac{1}{2}(1+3^{1+k-n})-\frac{9}{4}2^{-k}3^{k-n}}}{T^{\frac{1}{8}(1+9.3^{k-n})-\frac{3}{2}2^{-k}3^{k-n}}} \prod_{j=0}^{k-1} \log^{2^{3^{1+k-n}}/(j+2)}(k-j,k-j) \prod_{j=1}^{n-k-1} \log^{\frac{1}{4}3^{-j}}(n-j,k)$$
$$\text{for} \quad k < n, \tag{99}$$

$$\alpha_{n,n-1} = \alpha_{n,n}, \tag{100}$$

$$g_{n,k}^3 \log(n,k) \prod_{j=1}^{k} \log^{\frac{21}{2}3^{k-n}/2^j}(k+1-j, k+1-j) \prod_{j=1}^{n-k-1} \log^{\frac{7}{4}(1/3^j)}(n-j,k)$$
$$= \frac{T^{\frac{9}{8}(3+7.3^{k-n})-\frac{21}{8}2^{-k}3^{k-n}}}{R^{\frac{3}{2}(3+7.3^{k-n})-\frac{63}{4}2^{-k}3^{k-n}}} \quad \text{for} \quad k < n, \tag{101}$$

and
$$g_{n,n} \log(n,n) \prod_{j=1}^{n-1} \log^{3/2^{j+1}}(n-j, n-j) = \frac{T^{3(1-2^{-n})}}{R^{4-\frac{9}{2}2^{-n}}}. \tag{102}$$

These results can in fact be obtained by formally balancing (33) and (34) in each layer in a manner similar to the non-rotating case. There are, for the nth mode, $n+1$ sections to be considered, in the manner of table 1. In any event, having obtained these relations, the order of $F_{N,k}$ can be readily estimated as $T^{\frac{1}{2}}/(\alpha_{N,k}, g_{N,k})$. To maximize $F_{N,k}$, we have to vary the $b_{n,k}$'s, as well as the functions $\hat{\theta}_{n,k}$ and $\tilde{\theta}_{n+1,k}$, etc.

Upon substituting (99)–(102) into the expression for F, and considering specifically $k \leqslant N-1$, we have

$$K_{N,k} = \frac{1 - \sum\limits_{n=1}^{N} 2b_{n,k}^4 b_{n-1,k} \langle \tilde{\theta}_{n,k}^2 \rangle - \sum\limits_{n=k+1}^{N} 2b_{n,k}^{-7} \langle \hat{\theta}_{n,k}'^2 \rangle}{2 \sum\limits_{n=1}^{N} b_{N,k}(\alpha_{n,k} g_{n,k})/(\alpha_{N,k} g_{N,k}) \langle (1 - \hat{\omega}_{n,k} \hat{\theta}_{n,k} - \tilde{\theta}_{n+1,k}^2)^2 \rangle}, \tag{103}$$

with
$$F_{N,k} = K_{N,k} \frac{T^{\frac{1}{2}} b_{N,k}}{g_{N,k} \alpha_{N,k}}, \tag{104}$$

where $\hat{\omega}_{n,k}$ is given by (89) and (85) has been used to obtain the $\tilde{\theta}_{n+1,k}^2$ term in the denominator.

For fixed α's and g's (i.e. fixing $b_{n,k}$'s), the order of F is determined to be as large as possible within the limit of having only N modes. Its maximization with respect to ω and θ now means the maximization of $K_{N,k}$ only. For this purpose, it is necessary to set $\delta K_{N,k} = 0$ by varying the $\hat{\theta}_{n,k}$'s and the $\tilde{\theta}_{n+1,k}$'s. Doing this will result in only $2N-k$ equations, the additional equations must then be found by the formal limit of (33) and (34). For $k < n \leqslant N$, therefore, setting $\delta K_{N,k} = 0$ by varying $\hat{\theta}_{n,k}$ and $\tilde{\theta}_{n+1,k}$ gives

$$b_{n,k}^{-7} \frac{d^2 \hat{\theta}_{n,k}}{d\zeta_{n,k}^2} + K_{N,k} b_{N,k} \frac{\alpha_{n,k} g_{n,k}}{\alpha_{N,k} g_{N,k}} (1 - \hat{\omega}_{n,k} \hat{\theta}_{n,k} - \tilde{\theta}_{n+1,k}^2) \hat{\omega}_{n,k} = 0, \tag{105}$$

$$b_{n+1,k}^4 b_{n,k} \tilde{\theta}_{n+1,k} - 2K_{N,k} b_{N,k} \frac{\alpha_{n,k} g_{n,k}}{\alpha_{N,k} g_{N,k}} (1 - \hat{\omega}_{n,k} \hat{\theta}_{n,k} - \tilde{\theta}_{n+1,k}^2) \tilde{\theta}_{n+1,k} = 0. \tag{106}$$

For $n = N$, (106) is identically satisfied because $\tilde{\theta}_{N+1} \equiv 0$, while (105) now reduces to
$$b_{N,k}^{-7} \frac{d^2 \hat{\theta}_{N,k}}{d\zeta_{N,k}^2} + K_{N,k} b_{N,k} (1 - \hat{\omega}_{N,k} \hat{\theta}_{N,k}) \hat{\omega}_{N,k} = 0, \tag{107}$$

satisfying
$$\hat{\theta}_{N,k}(0) = 0; \quad \hat{\theta}_{N,k}(\zeta_{N,k} \to \infty) = \zeta_{N,k}^{-1}. \tag{108}$$

For $n \neq N$, as long as $\tilde{\theta}_{n+1,k} \neq 0$, (105) and (106) can be combined to yield

$$2b_{n,k}^{-7} \frac{d^2 \hat{\theta}_{n,k}}{d\zeta_{n,k}^2} + b_{n+1}^4 b_{n,k} \hat{\omega}_{n,k} = 0 \quad (N > n > k). \tag{109}$$

As $\zeta_{n,k} \to \infty$, however, $1-\hat{\omega}_n\hat{\theta}_n \to 0$ and so does $\tilde{\theta}_{n+1}$, so that (109) cannot hold for very large values of $\zeta_{n,k}$, i.e. $\hat{\theta}_{n,k}$ must now satisfy (109), with

$$\hat{\theta}_{n,k}(0) = 0, \tag{110a}$$

but the solution must merge (with a continuous first derivative) with the solution

$$1-\hat{\omega}_{n,k}\hat{\theta}_{n,k} = 0 \tag{110b}$$

numerically. Now, of the $4N$ unknowns, $\{\hat{\omega}_n, \hat{\theta}_n, \tilde{\omega}_n, \tilde{\theta}_n\}_1^N$, the set $\{\hat{\omega}_n, \tilde{\omega}_n\}_1^N$ has already been determined by (89) and (85). Of the remaining $2N$ unknown functions, (107) and (109) provide altogether $2N-2k$ constraints for

$$k+1 \leqslant n \leqslant N.$$

Another $2k$ constraints must now be sought directly from (33) and (34) in the $\zeta_{n,k}$ layer for $1 \leqslant n \leqslant k$:

$$-\frac{1}{RF_N}\alpha^4_{n+1,k}\tilde{\theta}_{n+1,k} + 2(1-\hat{\omega}_{n,k}\hat{\theta}_{n,k}-\tilde{\theta}^2_{n+1,k})\,\tilde{\theta}_{n+1,k} = 0, \tag{111}$$

$$-\frac{\alpha^2_{n,k}}{RF_N}\frac{T^2}{\alpha^6_{n,k}g^2_{n,k}\log(n,k)}D^2\hat{\theta}_{n,k} + D^2(1-\hat{\omega}_{n,k}\hat{\theta}_{n,k}-\tilde{\theta}^2_{n+1,k})\,\hat{\omega}_{n,k} = 0. \tag{112}$$

Equation (95) now yields, as long as $\tilde{\theta}_{n+1} \neq 0$,

$$\frac{d^2}{d\zeta^2_{n,k}}(-2\hat{\theta}_{n,k} + b^4_{n,k}b^4_{n+1,k}\hat{\omega}_{n,k}) = 0 \quad (1 \leqslant n \leqslant k), \tag{113a}$$

satisfying
$$\hat{\theta}_{n,k}(0) = 0, \tag{113b}$$

which must merge with

$$1-\hat{\omega}_{n,k}\hat{\theta}_{n,k} = 0 \quad \text{as} \quad \zeta_{n,k} \to \infty \tag{113c}$$

as $\tilde{\theta}_{n+1,k} \to 0$. Here, only the continuity of $\hat{\theta}_{n,k}$ is required, because (113*a*) is really algebraic. Having thus determined $\{\hat{\theta}_{n,k}\}_1^N$, the remaining functions $\{\tilde{\theta}_{n,k}\}_1^N$ are now simply given by

$$1-\hat{\omega}_{n,k}\hat{\theta}_{n,k} - \tilde{\theta}^2_{n+1,k} = 0. \tag{114}$$

By appropriate scaling, these equations will be reduced in the appendix to several typical systems free of any parameters. Using those results, $K_{N,k}$ can then be expressed in terms of the b's alone. For convenience, let us now drop the k subscript, keeping in mind that $k < N$, we finally have

$$K_N = \left[1-\left(b_1^4 + \sum_{n=1}^{k}\gamma\frac{b^2_{n+1}}{b_n} + \sum_{k+1}^{N-1}\lambda\frac{b^3_{n+1}}{b_n} + 2\beta K_N^{\frac{3}{4}}b_N^{-1}\right)\right]\Big/\left[2b_N\frac{\sigma}{K_N^{\frac{1}{4}}/4b_N^2}\right],$$

or
$$M_+ = 2I_+K_N^{\frac{3}{4}} = b_N\left(1-b_1^4-\gamma\sum_1^k\frac{b^2_{n+1}}{b_n} - \lambda\sum_{k+1}^{N-1}\frac{b^3_{n+1}}{b_n}\right), \tag{115}$$

where I_+ is defined in (63), and γ, λ, β are constants to be defined in the appendix. Maximizing K_N, or M_+, with respect to the b_n's readily gives, by setting

$$\partial K_N/\partial b_j = 0 \quad \text{for} \quad 1 \leqslant j \leqslant N,$$

$$\left.\begin{aligned}
&-4b_1^3+\gamma\frac{b_2^2}{b_1^2}=0, \quad \frac{b_{n+1}^2}{b_n^2}-\frac{2b_n}{b_{n-1}}=0 \quad (2\leqslant n\leqslant k),\\
&\frac{2\gamma b_{k+1}}{b_k}-\frac{\lambda b_{k+2}^3}{b_{k+1}^2}=0, \quad -\frac{b_{n+1}^3}{b_n^2}+\frac{3b_n^2}{b_{n-1}}=0 \quad (k+2\leqslant n\leqslant N-1),\\
&\left(1-b_1^4-\gamma\sum_1^k\frac{b_{n+1}^2}{b_n}-\lambda\sum_{k+1}^{N-1}\frac{b_{n+1}^3}{b_n}\right)-\lambda b_N\frac{3b_N^2}{b_{N-1}}=0.
\end{aligned}\right\} \tag{116}$$

After some straightforward though rather involved algebraic manipulation, (116) is solved to yield

$$b_{1,k}^4=(3^{N-k}2^{k+1}+2^{k+1}-3)^{-1}, \tag{117}$$

and

$$b_{n+1,k}=2^k\left(\frac{1}{\gamma}\right)^{1-2^{-n}}b_1^{4-3.2^{-n}} \quad (1\leqslant n\leqslant k), \tag{118}$$

$$b_{n+1,k}=3^{\frac{1}{2}(n-k-\frac{3}{2})+\frac{1}{4}3^{1+k-n}}2^{\frac{1}{2}[k+2+(k-2)3^{k-n}]}(1/\gamma)^{1-2.2^{-k}+2^{-k}3^{k-n}}(1/\gamma)^{\frac{1}{2}(1-3^{k-n})}$$
$$\times b_1^{2+3^{k-n}(2-3.2^{-k})} \quad (k+1\leqslant n\leqslant N-1).$$

Direct substitution then results, for $k<N$, in

$$K_{N,k}=I_+^{-\frac{4}{3}}3^{2[N-k-\frac{3}{2}+\frac{9}{2}3^{1+k-N}]}2^{2[k+2+(k-2)3^k-N]}\left(\frac{1}{\gamma}\right)^{\frac{4}{3}[1-2.2^{-k}+2^{-k}3^{1+k-N}]}\left(\frac{1}{\lambda}\right)^{\frac{2}{3}[1-3^{1+k-N}]}$$
$$\times(3^{N-k}2^{k+1}+2^{k+1}-3)^{-2-3^{1+k-N}(\frac{2}{3}-2^{-k})} \quad (k<N), \tag{119}$$

and

$$F_{N,k}=K_{n,k}^{2[4(1-2^{-k})-2k]3^{k-N}}\frac{R^{1+2.3^{k-N}-2^{-k}3^{1+k-N}}}{T^{\frac{1}{2}(1+3^{1+k-N})-2.2^{-k}3^{k-N}}}$$
$$\times\left(\log\frac{R^{\frac{3}{2}}}{T}\right)^{2(1-2^{-k})3^{k-N}}\prod_{j=0}^{N-k-1}\log^{3^{-j-1}}\left[\left(\frac{T}{R^{\frac{1}{3}}}\right)^{\frac{3}{4}+\frac{15}{4}3^{k+j-N}}\left(\frac{R^{\frac{3}{2}}}{T}\right)^{5.2^{-k}3^{j+k-N}}\right]$$
$$(k<N). \tag{120}$$

While these results are formally valid for any N, as long as $k<N$, the case $k=N$ can be easily obtained by some minor modification. This is the range where the rotation is so strong that, even for the Nth mode, $\alpha_{N,N}\gg F_{N,N}\gg 1$ holds. The mathematical manifestation of this is that there will be no $\langle(d\theta/dz)^2\rangle$ contribution to F. In fact, (103) reflects this rather clearly by adopting the convention $\sum\limits_{N+1}^{N}=0$. In addition, (105) and (106) will no longer hold, since the condition $k=N<n$ under which these were derived cannot be met. In its place, (111) and (112) will now hold for all $1\leqslant n\leqslant N$. The only necessary change here, then, is the equation for $\hat{\theta}_{N,N}$, which now becomes, from (112) with $\tilde{\theta}_{N+1,N}\equiv 0$,

$$-(K_{N,N}b_{N,N}^4)^{-1}D^2\hat{\theta}_{N,N}+D^2(1-\hat{\omega}_{N,N}\hat{\theta}_{N,N})\,\hat{\omega}_{N,N}=0, \tag{121}$$

satisfying

$$\hat{\theta}_{N,N}(0)=0 \quad\text{and}\quad \hat{\theta}_{N,N}(\zeta_{N,N}\to\infty)=\zeta_{N,N}^{-1}. \tag{122}$$

Appropriate scaling as shown in the appendix again reduces, dropping the second subscript for simplicity, to

$$K_N = \frac{1 - b_1^4 - \gamma \sum_1^{N-1} b_{n+1}^2/b_n}{2I_- b_N (K_N^{\frac{1}{2}} b_N^2)^{-1}},$$

or
$$M_- = 2I_- K_N^{\frac{1}{2}} = b_N\left[1 - b_1^4 - \gamma \sum_1^{N-1} b_{n+1}^2/b_n\right]. \tag{123}$$

The maximizing conditions corresponding to (116) now take the form

$$\left.\begin{aligned} -4b_1^3 + \gamma\frac{b_2^2}{b_1} = 0, \quad -\frac{b_{n+1}^2}{b_n^2} + \frac{2b_n}{b_{n-1}} = 0 \quad (2 \leqslant n \leqslant N-1),\\ \left(1 - b_1^4 - \gamma \sum_1^{N-1} \frac{b_{n+1}^2}{b_n}\right) - 2\gamma \frac{b_N^2}{b_{N-1}} = 0. \end{aligned}\right\} \tag{124}$$

Straightforward manipulation now yields

$$b_{1,N}^4 = (4.2^N - 3)^{-1}, \tag{125}$$

which is the same as (117) with $k = N$. Furthermore, in general,

$$b_{n+1,N} = 2^{n-\frac{3}{2}+\frac{3}{2}.2^{-n}}\left(\frac{1}{4.2^N - 3}\right)^{1-\frac{3}{4}.2^{-n}}. \tag{126}$$

Direct substitution now formally yields

$$K_{N,N} = \frac{1}{4I_-^2}\, 2^{4n-3+3.2^{1-N}}\left(\frac{1}{4.2^N-3}\right)^{4-3.2^{-N}}, \tag{127}$$

and
$$F_{N,N} = K_{N,N}\left(\frac{R^{\frac{2}{3}}}{T}\right)^{2(1-2^{-N})} \prod_{j=0}^{N-1} \log^{1/2^j}\left(\frac{T^2}{\alpha_{N-j,N-j}^8 g_{N-j,N-j}^2}\right), \tag{128}$$

or
$$F_{N,N} = K_{N,N}\, 2^{4(1-2^{-N})-2N}\left(\frac{R^{\frac{2}{3}}}{T}\log\frac{R^{\frac{2}{3}}}{T}\right)^{2(1-2^{-N})}. \tag{129}$$

Note that, for $N = 1$, (120) and (129) are reduced, respectively, to (68) and (77), as they should. Furthermore, while $K_{N,N}$ cannot be obtained from (119) by setting $k = N$, the order of $F_{N,N}$ can indeed be shown to be the same as that in (120) by taking $k = N$. We shall use this observation in §5.

5. The determination of N

While the solutions obtained in §4 are subject to the constraints of having N modes, the number N itself is as yet undetermined. There are really two aspects of this problem. To begin with, the formal solutions obtained above must satisfy certain assumptions made in the asymptotic analysis. If at some point the increase of N beyond a certain number violates any of those assumptions, it will then determine the maximum number of modes possible. On the other hand, if no such violations take place, N may then be allowed to increase indefinitely. As it turns out, for a fixed value of R, the total number of modes allowed is in fact finite as long as $T < O(R^{\frac{4}{3}})$, though N will be allowed to increase from 1 to 2 to 3, etc., as T increases. For $T > O(R^{\frac{4}{3}})$, however, N will be allowed to increase indefinitely depending on R and T. We shall proceed to show this in the present section.

Throughout the analysis, we have only said that the subscript k represents a certain relation between the parameters R and T. Such a relation was never explicitly determined. Having now obtained the asymptotic forms of the various quantities, table 1 now enables the determination of this relationship for each k. That is, in the k section, the results must satisfy

$$F_{k,k} \leqslant \alpha_{k,k} \quad \text{and} \quad F_{k+1,k} \geqslant \alpha_{k+1,k}. \tag{130}$$

If the exact forms for F and α are used, it is possible to define the relationship exactly. On the other hand, by using just the order-of-magnitude estimate, the algebra is much simplified and gives sufficient bounds within the context of asymptotic analysis. Following this path, then, and neglecting the logarithmic order, (130) requires that the kth section represents the requirement

$$R^{(4\cdot 2^k-3)/(3.2^k-2)} \leqslant T \leqslant R^{(8\cdot 2^k-3)/(6.2^k-2)}. \tag{131}$$

Note that the upper limit is the same as the lower limit when k is replaced by $k+1$ in the latter expression. This simply means, of course, that the lower limit of the $(k+1)$th section coincides with the upper limit of the kth section.

Now, for a given k, suppose a solution having N modes has been found, the existence of a solution with $N+1$ modes will then depend on

$$F^{-1}_{N+1,k} \leqslant T^{\frac{1}{2}}/\alpha^3_{N+1,k} \leqslant F^{-1}_{N,k} \tag{132}$$

being satisfied (see figure 1). This, to the same approximation as in (131), in turn implies

$$\begin{aligned} R_B &\equiv R^{(12.2^k3^{N-k}+20.2^k-30)/(9.2^k3^{N-k}+15.2^k-20)} \\ &\leqslant T \leqslant R^{(4.2^k3^{N-k}-4.2^k+6)/(3.2^k3^{N-k}-3.2^k+4)} \equiv R^C. \end{aligned} \tag{133}$$

In addition to (133), we must also have

$$F_{N+1,k} \geqslant \alpha_{N+1,k}, \tag{134}$$

which requires that

$$T \leqslant R^{(12.2^k3^{N-k}+4.2^k-6)/(9.2^k3^{N-k}+3.2^k-4)} \equiv R^A. \tag{135}$$

Thus, a solution having $N+1$ modes exists only if both (133) and (135) are satisfied. This, in turn, depends on whether $N > k$ or $N = k$, i.e. whether the N-mode solution is such that $F_{N,k} > \alpha_{N,k}$ or $F_{N,k} < \alpha_{N,k}$ (which happens only if $N = k$), respectively. Table 1 shows clearly that the case $N = k+1$ is the most crucial step, as that will always be the first time that $F_{N,k} > \alpha_{N,k}$ happens. Using this value for N in (133) and (135) immediately shows that $C > A$ for all k (i.e. once a solution has been obtained that $F > \alpha$, it will not be possible to introduce higher modes). On the other hand, for $N = k$, (133) and (135) now become, respectively,

$$R^{(16.2^k-15)/(12.2^k-10)} \leqslant T \leqslant R^{\frac{3}{2}}, \tag{136}$$

which is seen to be superfluous when compared with (131). Furthermore, the lower bound of (131) clearly has $R^{\frac{4}{3}}$ as its lowest upper bound. This implies, therefore, that, for $T < O(R^{\frac{4}{3}})$, the maximizing solution can have at most finitely many modes, the last one being that $F_{N,k} \geqslant \alpha_{N,k}$ for the first time with respect to N. This, of course, happens at $N = k+1$. In other words, the kth section (i.e. (131)) will have a maximizing solution having $k+1$ modes. On the other hand, (131) shows that the range $T \geqslant O(R^{\frac{4}{3}})$ always lies in the (k, k) mode for whatever value of k (as long as $T \geqslant O(R^{\frac{4}{3}})$); $F_{N,N} < \alpha_{N,N}$ always holds. In other words, for this range, the maximizing solution is allowed to have indefinitely many modes (N remains arbitrary).

We must now turn our attention to this range, where F is given by (127) and (129). To determine N, we let N vary, and seek the value of N for which

$$\partial F_{N,N}/\partial N = 0.$$

Writing

$$\mu = 2^{-N},$$

and treating it as a continuous variable, $\partial F_{N,N}/\partial N = 0$ is now equivalent to $d(\log F_{N,N})/d\mu = 0$, i.e.

$$0 = F_{N,N}^{-1}\frac{\partial F_{N,N}}{\partial \mu} = \left[-\frac{4}{\mu} + 6\log 2 - 3\log \mu + 3\log(4-3\mu) + \frac{4}{\mu} + \frac{12}{4-3\mu} - 3 \right.$$
$$\left. - \frac{9\mu}{4-3\mu} - 4\log 2 + \frac{2}{\mu} - 2\log\left(\frac{R^{\frac{3}{2}}}{T}\log\frac{R^{\frac{3}{2}}}{T}\right)\right] = 0.$$

For large N, $\mu \to 0$, so that the dominant terms are

$$2/\mu - 2\log(R^{\frac{3}{2}}/T) = 0; \tag{137}$$

i.e. N will in general depend on R and T as

$$N = (\log 2)^{-1}\log \mu^{-1} = (\log 2)^{-1}\log\log(R^{\frac{3}{2}}/T). \tag{138}$$

Substituting this value into (127) and (129) then gives, in the limit of large R and T with $T \geqslant O(R^{\frac{4}{3}})$,

$$F \simeq (2^9 I_{-}^{2})^{-1} R^3/T^2. \tag{139}$$

6. Discussion

Qualitatively, the asymptotic analysis indicates that it is best to divide the parameter space into three different regions. For a weakly rotating system ($T \leqslant O(R)$), the rotational constraint is not felt, and convection takes place in much the same way as a non-rotating system. For the free–free boundary geometry, it is estimated (Chan 1971) that the solution has only one horizontal mode, with $\alpha \sim O(R^{\frac{1}{4}})$ and the Nusselt number proportional to $O(R^{\frac{1}{3}})$. For a moderately large rotational constraint ($O(R) \leqslant T \leqslant O(R^{\frac{4}{3}})$), the solution may have a multi-α structure, depending on the strength of the rotation. For any given R and T in this range, however, there are only finitely many horizontal modes, no matter how much R and T are increased. The detailed structure will depend on finer subdivision in this range. The criterion here is that the solution will have as many modes as are necessary for $F_N \geqslant O(\alpha_N)$ to be achieved. This is always possible at some finite value for N, though N increases with T. Once this purpose is accomplished, however, it will not be possible to increase the total number of horizontal modes any further. For a strongly rotating system ($O(R^{\frac{4}{3}}) \leqslant T \leqslant O(R^{\frac{3}{2}})$), the suppression of convective motion is so strong that $F_N \geqslant O(\alpha_N)$ can never be achieved. While it may be said that the system nonetheless struggles toward this aim, the net result is merely an indefinite increase of the total number of modes as R and T increase together while staying within this region of the parameter space. For an even stronger rotation rate ($T \geqslant O(R^{\frac{4}{3}})$), the suppression is total, so that no convective heat transport is possible, in agreement with the linear theory.

It is believed (Veronis 1959) that, for a high Prandtl number flow, the effect of the rotational constraint is always to suppress convective motion so that the transport of heat by convection is always less effective than in the non-rotating case. The increase of horizontal modes, to the extent that it may reflect the qualitative feature of the real flow, on the other hand, is associated with the breakdown of the flow into smaller eddies giving rise to an increase in heat transport. If the effect of a rotational constraint is in fact to suppress convection, one may feel intuitively that, the stronger the rotation, the more it will tend to suppress small eddies, and therefore fewer modes will be allowed, just the opposite of the asymptotic result. This apparent contradiction can in fact be easily resolved by the following consideration. Essentially, the Malkus hypothesis assumes that the flow will always organize itself so as to give rise to an optimal heat transport. This is done first by a choice of the length scales of the motion, then by a choice of the shortest scale (i.e. N). When the rotation is only moderate, the suppression of heat transport is not very effective, though the heat flux is somewhat less than the non-rotating value. Had the flow been allowed to have a very small scale of motion (i.e. higher N), it would have caused the heat transport to be in fact higher than in the non-rotating case. Therefore, the value for N is always limited. On the other hand, when the rotation is very strong, the suppression of heat transport is so strong that, no matter how many modes the flow breaks down into, the resultant heat flux is still less than the non-rotating value. As a result, N is allowed to increase indefinitely in its effort to remove the rotational constraint.

As was mentioned above, the non-rotating Nusselt number is estimated to be proportional to $R^{\frac{1}{3}}$. Given that the effect of the rotational constraint on an infinite Prandtl number fluid is to suppress heat transport, the Nusselt number must not be allowed to vary more rapidly than $R^{\frac{1}{3}}$. One sees readily that this is in fact the case. From the results obtained, it is seen that, for whatever value of k, the maximum heat flux is attained when $N = k+1$ and T is at the lower bound of (131). At these values, (120) immediately shows that $F \sim R^{\frac{1}{3}}$, and for any slight increase of T within that k section, F will of course be less than $O(R^{\frac{1}{3}})$.

This suggests another interpretation of the role of the total number of modes, namely that the flow will organize itself by having as many modes as possible so as to tend to approach the non-rotating case (i.e. to try to remove the effect of the rotational constraint). In his study of finite-amplitude convection, Veronis (1959) concluded that the flow, through the nonlinear advective terms, tends to generate internal motions that counteract the rotational constraint. For an infinite Prandtl number fluid, the effects of that nonlinear interaction have been neglected. Nonetheless, within the context of a quasi-linear model, the flow remains capable of rearranging itself so as to counteract the imposed constraint, though by a completely different mechanism. This generation of internal motion manifests itself in the form of having more and more modes as the constraint gets stronger and stronger.

In this regard it should be pointed out that the asymptotic solutions were obtained under the assumption that $|g| \ll 1$ for the case $F \geqslant \alpha$. This is necessary to obtain, e.g., the balance as in (55). Such a requirement, as it turns out, imposes a further condition that

$$T \leqslant R^{(32.2^k-21)/(24.2^k-14)}, \tag{140}$$

which lies somewhat below the upper bound of (131). What this means is that there are sections within the range $O(R) \leqslant T \leqslant O(R^{\frac{4}{3}})$ for which the $F_{k+1,k}$ formula does not apply. We cannot say for sure at this time whether the non-applicability of the asymptotic results there is a consequence of the formal procedure used, and some other ordering process may extend the present result to these ranges, or whether it is a consequence of having two interacting scales of motion (as suggested by Rossby 1970). It is believed, however, that the conclusion drawn above about the nature of N should remain valid throughout. In any event, the same difficulty is not encountered for the range

$$O(R^{\frac{4}{3}}) \leqslant T \leqslant O(R^{\frac{3}{2}}).$$

It is also interesting to note that, if R and T are made infinite by making $d \to \infty$ and holding everything else fixed, T and R satisfy the relation $T \sim O(R^{\frac{4}{3}})$. Dimensionality argument may then suggest that the total dimensional heat flux in that case should be independent of d. This is in fact the case, since

$$F \sim R^3/T^2 \sim d^9/d^8 \sim d.$$

In the non-rotating case, the same argument suggests a Nusselt number dependence of $R^{\frac{1}{3}}$. Though no hints as to the functional dependence can be suggested in the rotating case by similar arguments, it is nonetheless interesting to note that $F \sim d$ for large T, and R still holds as long as $T \geqslant O(R^{\frac{4}{3}})$.

While the nature of the boundary condition studied renders it impossible to make detailed comparison with any experimental work, it is nonetheless instructive to compare the present result with the work of Rossby (1970). Rossby observed that, for fixed R, the heat flux first increases above its non-rotational value as T is increased, until it reaches a maximum, then decreases again as T increases. He thus raised the question of whether or not it is possible that the first observable effect of the rotation may be the interaction of the thermal layer with the Ekman layer, thereby making the boundary condition free so that the fluid ends up by convecting more heat than in the non-rotating case. Our study seems to rule out such a possibility, since the first observable effect of the rotational constraint comes in when $T \geqslant O(R)$ for which $F \sim R^{\frac{2}{3}}/T^{\frac{1}{3}}$, which is less than $R^{\frac{1}{3}}$, as T increases beyond $O(R)$. It seems more likely that any initial increase of the heat flux with respect to T may be due to some nonlinear interaction between the fluctuating quantities, which is neglected here. In his study on subcritical instability in a rotating fluid, Veronis (1966) explained that subcritical steady convection can exist provided that the Prandtl number is less than a certain value. This is mathematically equivalent to the inclusion of the self-interacting terms of the momentum equation. On the other hand, the infinite Prandtl number assumption is equivalent to the mean-field approach (Herring 1963), where only the interaction between the mean and the fluctuating fields is included. In this case, the heat transport by convection is achieved by the distortion of the mean temperature profile by the fluctuating quantities. With a rotational constraint, some of the energy released from the mean profile is used up against the balance of the thermal wind; therefore the effect of a rotational constraint is to transport less heat upward. This is indeed the case here, as the balance in the transition layer of order $T^{\frac{1}{2}}/\alpha^3$ is achieved by a balance of the $TD^2\omega$ and the $\nabla_1^2\theta$ term (i.e. the vertical variation of the velocity component and the horizontal variation of the temperature gradient). Consequently, the heat flux is always decreasing with increasing T.

On the other hand, we do find evidence suggesting that, for a strongly rotating system, the effect of the Ekman layer is indeed to make the boundary conditions look free. For $T \geqslant O(R^{\frac{4}{5}})$, the thermal layer is of order $1/F \sim T/R^{\frac{3}{2}}$, which is much thicker than the Ekman layer $O(T^{-\frac{1}{4}})$. In this case, even if the boundaries are rigid, we can still treat them as free in the thermal layer. The rigid boundary condition $\partial\omega/\partial z = 0$ can now be satisfied by the balance

$$(D^6 + TD^2)\,\omega = 0,$$

with the matching condition $\omega \rightarrow \zeta$ as $\zeta \rightarrow \infty$. This would indeed cause the heat transport to be higher than it might have been for a rigid boundary geometry, but not high enough to surpass that of the non-rotating case.

Finally, one may ask how the asymptotic solutions will change if the more realistic case of rigid boundaries is studied. Some preliminary results indicate that, at least for the single-α case, the maximizing wavenumber is the same as that of the non-rotating case, giving rise to a Nusselt number proportional to $R^{\frac{3}{10}}$ with some additional factor depending on T in a logarithmic manner, so that the effect of T on F is in fact quite weak here. Beyond that, it seems that, since the solution for the non-rotating case allows N to increase without bound for large R, the same may be true for the rotating case as well. It also seems rather likely that, especially within the context of the maximum principle, whenever T is large enough so that the Ekman layer becomes thinner than the thermal layer, the fluid will behave as if the boundaries are free, thereby transporting heat somewhat more effectively. In this sense, then, the present result may be valid also for rigid boundaries when $T^{\frac{1}{4}} \ll 1/F$.

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Appendix

The differential equation (107) for $\hat{\theta}_N$ and its boundary conditions (108) suggest the normalization

$$\hat{\theta}_{N,k} = (b^8_{N,k}K_{N,k})^{\frac{1}{4}}f, \quad \zeta_{N,k} = [1/(b^8_{N,k}K_{N,k})]^{\frac{1}{4}}\zeta. \tag{A 1}$$

Together with (89), (107) and (108) then reduce to

$$d^2f/d\zeta^2 + (1-\zeta f)\zeta = 0, \quad f(0) = 0, f(\infty) = \zeta^{-1}. \tag{A 2}$$

It can readily be seen that (A 2) is the same as (59), so that

$$f = \tfrac{1}{2}\zeta\int_0^1 \exp\{-\tfrac{1}{2}\zeta^2 t\}(1-t^2)^{\frac{1}{4}}\,dt. \tag{A 3}$$

Similarly, it is seen that, for $k < n < N$,

$$\hat{\theta}_{n,k} = [(b^4_{n+1,k}b^8_{n,k})/2]^{\frac{1}{4}}g, \quad \zeta_{n,k} = [2/(b^4_{n+1,k}b^8_{n,k})]^{\frac{1}{4}}\eta. \tag{A 4}$$

Equation (109), using (89), becomes

$$d^2g/d\eta^2 + \eta = 0, \tag{A 5}$$

which must satisfy the boundary condition $g(0) = 0$ and merge numerically with η^{-1} with a continuous first-order derivative; i.e.

$$g(\eta) = \begin{cases} -\tfrac{1}{2}\eta^2 + (3/2\sqrt[3]{2})\,\eta, & 0 \leqslant \eta \leqslant \sqrt[3]{4}, \\ \eta^{-1}, & \eta \geqslant \sqrt[3]{4}. \end{cases} \tag{A 6}$$

Likewise, for $1 \leqslant n \leqslant k$, (89) and the scaling

$$\hat{\theta}_{n,k} = [(b^4_{n,k}b^4_{n+1,k})/2]^{\frac{1}{4}}h, \quad \zeta_{n,k} = [2/(b^4_{n,k}b^4_{n+1,k})^{\frac{1}{4}}]\,\xi \tag{A 7}$$

reduce (113*a*) to

$$(d^2/d\xi^2)(-h+\xi) = 0, \tag{A 8}$$

which must satisfy $h(0) = 0$ and merges numerically with the function ξ^{-1}. Here, a discontinuity in its first derivative is allowed, since $(d\theta/dz)^2$ for these modes does not enter in the formula for the Nusselt number. Presumably, this signifies an internal boundary layer, since (111) and (112) really is a system of equations with a small parameter. For our purpose, however, we need not concern ourselves with the structure of this boundary layer. At any rate,

$$h(\xi) = \begin{cases} \xi, & \xi \leqslant 1, \\ \xi^{-1}, & \zeta \geqslant 1. \end{cases} \tag{A 9}$$

By direct substitution into (103), (115) follows, where the constants are given by

$$\sigma = \int_0^\infty (1-\zeta f)^2 d\zeta, \quad \beta = \int_0^\infty \left(\frac{df}{d\zeta}\right)^2 d\zeta, \tag{A 10}$$

$$I_+ = \sigma + \beta = \int_0^\infty \left[\left(\frac{df}{d\zeta}\right)^2 + (1-\zeta f)^2\right] d\zeta = 1{\cdot}063, \tag{A 11}$$

$$\gamma = 2\sqrt{2}\int_0^\infty (1-\xi h)\,d\xi = \frac{4\sqrt{2}}{3}, \tag{A 12}$$

$$\lambda = 2^{\frac{5}{4}}\int_0^\infty (1-\eta g)\,d\eta + 2^{\frac{1}{4}}\int_0^\infty \left(\frac{dg}{d\eta}\right)^2 d\eta = 2^{\frac{11}{12}} + \tfrac{5}{6}.2^{\frac{1}{4}}. \tag{A 13}$$

Finally, for $k = N$, using (89) and

$$\hat{\theta}_{N,N} = (K_{N,N}b^4_{N,N})^{\frac{1}{2}}q, \quad \hat{\zeta}_{N,N} = (K_{N,N}b^4_{N,N})^{-\frac{1}{2}}\zeta, \tag{A 14}$$

(121) and (122) become

$$D^2[-q+(1-\zeta q)\zeta] = 0, \tag{A 15}$$

satisfying $\qquad q(0) = 0 \quad \text{and} \quad q(\infty) = \zeta^{-1}.$ (A 16)

Hence, $\qquad q = \zeta/(1+\zeta^2),$ (A 17)

which is the same as (73). Consequently,

$$I_- = \int_0^\infty (1-\zeta q)^2 d\zeta = \tfrac{1}{4}\pi. \tag{A 18}$$

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